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The self-amalgamation of high distance Heegaard splittings is always efficient

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ABSTRACT

Let M be a compact orientable irreducible 3-manifold, F be an essential non-separating closed surface in M. We denote by $\eta(F)$ the open regular neighborhood of F. If $M-\eta(F)$ has a high distance Heegaard splitting, then M has a unique minimal Heegaard splitting up to isotopy.

1. Introduction

All 3-manifolds in this paper are assumed to be compact and orientable.

Let M be a 3-manifold. If there is a closed surface S which cuts M into two compression bodies V and W with $S=\partial_+W=\partial_+V$, then we say M has a Heegaard splitting, denoted by $M=V\cup_S W$; and S is called a Heegaard surface of M. Moreover, if the genus $g(S)$ of S is minimal among all Heegaard surfaces of M, then $g(S)$ is called the genus of M, denoted by $g(M)$. Specially, let M be a bounded 3-manifold, and $\mathcal{F}$ be a collection of boundary components of M. If $M=V\cup_S W$ is a Heegaard splitting such that $\mathcal{F}\subset\partial_-V$ or $\mathcal{F}\subset\partial_-W$, then $M=V\cup_S W$ is called a Heegaard splitting relative to $\mathcal{F}$. In this case, if $g(S)$ is minimal among all the Heegaard splittings of M relative to $\mathcal{F}$, then $g(S)$ is called the minimal genus of M relative to $\mathcal{F}$, denoted it by $g(M,\mathcal{F})$.

If there are essential disks $B\subset V$ and $D\subset W$ such that $\partial B=\partial D$ (resp. $\partial B\cap\partial D=\emptyset$), then $V\cup_S W$ is said to be reducible (resp. weakly reducible). Otherwise, it is said to be irreducible (resp. strongly irreducible). If there are essential disks $B\subset V$ and $D\subset W$ such that $|B\cap D|=1$, then $V\cup_S W$ is said to be stabilized. Otherwise, it is said to be unstabilized.

If a properly embedded surface F in a 3-manifold M is incompressible and not parallel to ∂M, then F is said to be essential. If a properly embedded separating surface F in M is compressible on both sides of F, then F is said to be bicompressible. If every compressing disk in one side of F intersects every compressing disk in the other side, then F is said to be strongly irreducible.

Let $M=V\cup_S W$ be a Heegaard splitting. The distance between two essential simple closed curves α and β on S, denoted by $d(\alpha,\beta)$, is the smallest integer $n\geqslant 0$ so there is a sequence of essential simple closed curves $\alpha_0=\alpha,\dots,\alpha_n=\beta$ on S

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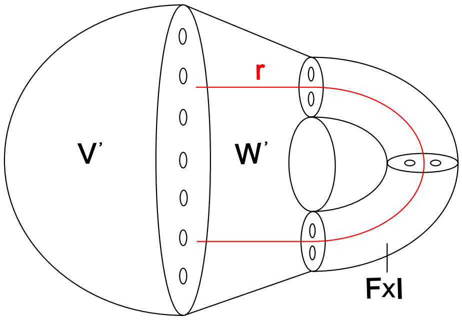

Fig. 1.

such that α_{i-1} is disjoint from α_i for $1 \leqslant i \leqslant n$. The distance of the Heegaard splitting $W \cup_S V$ is $d(S) = Min\{d(\alpha, \beta)\}$, where α bounds a disk in V and β bounds a disk in W. See [3].

Let M be an irreducible 3-manifold, F be an essential connected closed surface in M which cuts M into two 3-manifolds M_1 and M_2. If $M_i = V_i \cup_{H_i} W_i$ is a Heegaard splitting of M_i $(i = 1, 2)$, then M has a natural Heegaard splitting called the amalgamation of $V_1 \cup_{H_1} W_1$ and $V_2 \cup_{H_2} W_2$. See [14]. From this construction, we have $g(M) \leqslant g(M_1) + g(M_2) - g(F)$.

Suppose now F is an essential non-separating closed surface in M. Let $M' = M - int(F \times [0, 1])$, $F_1 = F \times \{0\}$ and $F_2 = F \times \{1\}$. If $V' \cup_{S'} W'$ is a Heegaard splitting of M' such that F_1 and F_2 lie in the same side of S', say in W', then there is a natural Heegaard splitting of M as follows.

Since W' is obtained by attaching some 1-handles to $\partial_- W' \times I$, we can take two unknotted arcs $a = \{a_0\} \times I$ and $b = \{b_0\} \times I$ in $\partial_- W' \times I$, where a_0 and b_0 lie in F, such that they are disjoint from all 1-handles in W'. Let c be an another unknotted arc in $F \times [0, 1]$, such that $r = a \cup b \cup c$ is a properly embedded arc in $W' \cup F \times [0, 1]$. See Fig. 1. Let $V = V' \cup N(r)$, $W = cl(M - V)$. It is easy to see that V and W are both compression bodies. The Heegaard splitting $V \cup W$ is said to be the self-amalgamation of $V' \cup_{S'} W'$. From this construction, we have $g(M) \leqslant g(M', F_1 \cup F_2) + 1$.

Kobayashi and Qiu prove that if M_1 and M_2 have high distance Heegaard splittings, then the minimal Heegaard splitting of the amalgamated 3-manifold of M_1 and M_2 along F is unique. See [4]. An extension of Kobayashi and Qiu's result was given by Lei and Yang. See [16]. Du, Lei and Ma have proved that if a closed 3-manifold M is the self-amalgamation of M', and M' has a high distance Heegaard splitting relative to $F_1 \cup F_2$, then the minimal Heegaard splitting is unique. See [1]. The main result in this paper is as follows:

Theorem 1. *Let M be an irreducible 3-manifold, F be an essential non-separating closed surface in M. Suppose $V' \cup_{S'} W'$ is a Heegaard splitting of M' with $d(S') > 2(g(M', F_1 \cup F_2) + 1)$. Then the minimal Heegaard splitting of M is unique up to isotopy, i.e. the self-amalgamation of $V' \cup_{S'} W'$, where $M' = M - int(F \times [0, 1])$, $F_1 = F \times \{0\}$ and $F_2 = F \times \{1\}$.*

2. Preliminary

Lemma 1. *([2,9]) Let $M = V \cup_S W$ be a Heegaard splitting, and F be an incompressible surface in M. Then either F can be isotoped to be disjoint from S or $d(S) \leqslant 2 - \chi(F)$.*

Lemma 2. *([13]) Let $V \cup_S W$ be a Heegaard splitting such that $d(S) > 2g(M)$. Then $V \cup_S W$ is the unique minimal Heegaard splitting of M up to isotopy.*

Lemma 3. *([8]) Let $M = V \cup_S W$ be a strongly irreducible Heegaard splitting, and $\mathcal{F}$ be a minimal separating system in M which cuts M into two manifolds M_1 and M_2. Then S can be isotoped so that*

(1) *each of $S \cap M_1$ and $S \cap M_2$ is incompressible, or*
(2) *one of $S \cap M_1$ and $S \cap M_2$, say $S \cap M_1$, is incompressible while all components of $S \cap M_2$ are incompressible except one bicompressible component,*
(3) *one of $S \cap M_1$ and $S \cap M_2$, say $S \cap M_1$, is incompressible while $S \cap M_2$ is compressible, and there is a Heegaard surface S' isotopic to S such that*
    (i) *$S' \cap M_1$ is compressible while $S' \cap M_2$ is incompressible, and*
    (ii) *S' is obtained by ∂-compressing S in M_2 only one time.*

Now, we give an outline proof to this lemma.

Proof. Let $\{H_1, H_2\} = \{W, V\}$. If each of $S \cap M_1$ and $S \cap M_2$ is incompressible, then Lemma 3(1) holds. If one of $S \cap M_1$ and $S \cap M_2$ is bicompressible, then, since $V \cup_S W$ is strongly irreducible, Lemma 3(2) holds. We may assume that

(1) one of $S \cap M_1$ and $S \cap M_2$ is compressible, and we may assume which is compressible in H_1, so $S \cap M_i$ is incompressible in $M_i \cap H_2$ for $i = 1, 2$;
(2) $S \cap M_i$ is incompressible in $M_i \cap H_2$ for $i = 1, 2$.

Since $\mathcal{F}$ is a collection of essential surfaces in M, H_1 and H_2 are non-trivial compression bodies. Let D be an essential disk of H_2 such that $|D \cap \mathcal{F}|$ is minimal among all essential disks in H_2. By Assumption (2), $|D \cap \mathcal{F}| > 0$. Furthermore, we may assume that

(3) S is a strongly irreducible Heegaard surface such that $|D \cap \mathcal{F}|$ is minimal among all Heegaard surfaces isotopic to S and satisfies Assumptions (1) and (2).

Let a be an outermost component of $D \cap \mathcal{F}$ on D. This means that a, together with an arc b on $\partial D (\subset S)$, bounds a disk B in D which lies in either $M_1 \cap H_2$ or $M_2 \cap H_2$ such that $B \cap \mathcal{F} = a$, and we may assume $D \subset M_2 \cap H_2$. By the minimality of $|D \cap \mathcal{F}|$, a is essential in $\mathcal{F}$, so B is a ∂-compressing disk of $S \cap M_1$ or $S \cap M_2$.

Now there are two cases:

Case 1. $S \cap M_1$ is compressible in $M_1 \cap H_1$.

By Assumption (2), $S \cap M_2$ is incompressible in $M_2 \cap H_2$.

Now let S' be the Heegaard surface of M obtained by ∂-compressing S along D. We denote by H_1' and H_2' the two components of $M - S'$. We may assume that $M_1 \cap H_1 \subset M_1 \cap H_1'$. Since $S \cap M_1$ is compressible in $M_1 \cap H_1$, $S' \cap M_1$ is compressible in $M_1 \cap H_1'$, and $S' \cap M_2$ is incompressible in $M_2 \cap H_2'$. Now if $S' \cap M_1$ is compressible in $M_1 \cap H_2'$, then Lemma 3(2) holds.

Suppose that $S' \cap M_1$ is incompressible in $M_1 \cap H_2'$. Then $S' \cap M_i$ is either incompressible or compressible in $M_i \cap H_1'$ but not bicompressible. Now $D \cap H_2'$ is an essential disk in H_2'. But $|D \cap H_2' \cap \mathcal{F}| = |D \cap \mathcal{F}| - 1$. This contradicts Assumptions (1)–(3).

Case 2. $S \cap M_2$ is compressible in $M_2 \cap H_1$, and $S \cap M_1$ is incompressible in $M_1 \cap H_1$.

By Assumption (2), $S \cap M_1$ is incompressible in $M_1 \cap H_2$. Hence $S \cap M_1$ is incompressible in M_1. Similarly, let S' be the Heegaard surface of M obtained by ∂-compressing S along D. We denote by H_1' and H_2' the two components of $M - S'$. We may assume that $M_1 \cap H_1 \subset M_1 \cap H_1'$. By Assumption (2), $S \cap M_2$ is incompressible in $M_2 \cap H_2$. Hence $S' \cap M_2$ is incompressible in $M_2 \cap H_2'$. If $S' \cap M_2$ is incompressible in $M_2 \cap H_1'$, then Lemma 3(3) holds.

Suppose that $S' \cap M_2$ is compressible in $M_2 \cap H_1'$. Since S' is also a strongly irreducible Heegaard surface, $S' \cap M_1$ is incompressible in $M_1 \cap H_2'$. But $|D \cap H_2' \cap \mathcal{F}| = |D \cap \mathcal{F}| - 1$. This contradicts Assumptions (1)–(3). □

Lemma 4. *Let M be a compact orientable irreducible 3-manifold, $V \cup_P W$ be a Heegaard splitting of M, Q be a properly embedded strongly irreducible bounded surface in M. Then either $d(P) \leqslant 2 - \chi(Q)$ or Q lies in an I-bundle of one component of ∂M.*

Tao Li proves the lemma when Q is a closed surface. In fact, his argument is also true for bounded surface. See [6].

Lemma 5. *([13]) Suppose P and Q are both Heegaard surfaces for the compact orientable 3-manifold M. Then either $d(P) \leqslant 2g(Q)$ or Q is isotopic to P or to a stabilization or ∂-stabilization to P.*

3. The proof of Theorem 1

Recall that M is an irreducible 3-manifold, F is an essential non-separating closed surface in M. Let $M' = M - int(F \times [0, 1])$, $F_1 = F \times \{0\}$ and $F_2 = F \times \{1\}$. $V' \cup_{S'} W'$ is a Heegaard splitting of M' with $d(S') > 2(g(M', F_1 \cup F_2) + 1)$.

Proof of Theorem 1. Case 1. F_1 and F_2 lie in different sides of S'. We may assume F_1 lies in V', and F_2 lies in W'.

Let $V \cup_S W$ be a minimal Heegaard splitting of M. Since F is essential, S cannot be isotoped to be disjoint from F. Let $M_1 = F \times [0, 1]$.

Case 1.1. $V \cup_S W$ is strongly irreducible.

Let $S_1 = S \cap M_1$, $S_2 = S \cap M'$. By Lemma 3, one of S_1 and S_2 is incompressible. If S_2 is incompressible, by Lemma 1, either S_2 is disjoint from S' or $d(S') \leqslant 2 - \chi(S_2)$. Since $\chi(S_2) \geqslant \chi(S)$, $d(S') > 2(g(M', F_1 \cup F_2) + 1) \geqslant 2g(M) = 2 - \chi(S) \geqslant 2 - \chi(S_2)$, S_2 is disjoint from S'. By Lemma 2.3 in [9], we have each component of S_2 is parallel to $F_1 \cup F_2$. Then, S can be isotoped to be disjoint from F, a contradiction. Hence, by Lemma 3, we may assume that S_1 is incompressible. So, there are only two cases:

Firstly, S_1 is incompressible, S_2 is compressible, say in $M' \cap V$ and incompressible in $M' \cap W$.

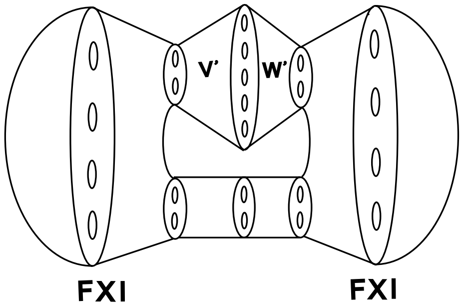

Fig. 2.

By Lemma 3, we get an incompressible surface S_2' after ∂-compressing S_2 in M' only one time. Since $\chi(S_2') \geqslant \chi(S)$, $d(S') > 2(g(M', F_1 \cup F_2) + 1) \geqslant 2g(M) = 2 - \chi(S) \geqslant 2 - \chi(S_2')$, by Lemma 1, S_2' is disjoint from S'. By Lemma 2.3 in [9], S_2' is parallel to $F_1 \cup F_2$. Then S can be isotoped to be disjoint from F, a contradiction.

Now we suppose that S_1 is incompressible, S_2 is bicompressible.

We may assume S_2 is connected. Otherwise, if it is not connected, then only one component of S_2 is bicompressible, other components of S_2 are incompressible, since $V \cup_S W$ is strongly irreducible. By the argument as above, any incompressible component of S_2 is parallel to $F_1 \cup F_2$, we can push it into M_1. After maximally compressing S_2 in $M' \cap V$ (resp. $M' \cap W$), we denote it by S_V (resp. S_W). By no nested lemma in [10], S_V and S_W are incompressible.

By the argument as above, we may assume each bounded component of S_V and S_W is ∂-parallel. Note that $S_2 \cap F_1 \neq \emptyset$ and $S_2 \cap F_2 \neq \emptyset$. If the bounded components of S_V (resp. S_W) are nested, since S_2 is connected, we will have $S_2 \cap F_1 = \emptyset$ or $S_2 \cap F_2 = \emptyset$, a contradiction. Hence, each bounded component of S_V is ∂-parallel and non-nested. So does S_W.

By Lemma 4, $d(S) \leqslant 2 - \chi(S_2) \leqslant 2 - \chi(S) = 2g(S) \leqslant 2(g(M', F_1 \cup F_2) + 1)$, a contradiction.

Case 1.2. $V \cup_S W$ is weakly reducible.

Since $V \cup_S W$ is weakly reducible, by [11], $M = V \cup_S W = (V_1 \cup_{P_1} W_1) \cup_{H_1} \cdots \cup_{H_{n-1}} (V_n \cup_{P_n} W_n)$, where each $V_i \cup_{P_i} W_i$ is strongly irreducible, each H_i is incompressible.

If there is H_i for some i, such that $H_i \cap (F_1 \cup F_2) \neq \emptyset$ after isotopies, by the proof as above, $H_i \cap M'$ is ∂-parallel. We can push it into M_1. Hence, for each i, $H_i \cap (F_1 \cup F_2) = \emptyset$. Since M_1 is an I-bundle, M' has a high distance Heegaard splitting, each component of H_i is parallel to $F_1 \cup F_2$. Then, let $H_1 = F_1 \cup F_2$, $V_1 \cup_{P_1} W_1 = M_1$, and $V'' \cup_{S''} W'' = M'$ is the amalgamation of $(V_2 \cup_{P_2} W_2) \cup_{H_2} \cdots \cup_{H_{n-1}} (V_n \cup_{P_n} W_n)$. Note that F_1 and F_2 lie in the same sides of S'', say in V''. Since $g(S) = g(S'') + 1$, we have $d(S') > 2(g(M', F_1 \cup F_2) + 1) \geqslant 2g(S) > 2g(S'')$. By Lemma 5, S'' is a ∂-stabilization of S'. Since F_1 and F_2 lie in different compression bodies of V' and W', and lie in the same compression body V'' or W'', $g(S'') \geqslant g(S') + g(F)$, we have $g(S) \geqslant g(S') + g(F) + 1 \geqslant g(M', F_1 \cup F_2) + 1$. Note that $g(S) \leqslant g(M', F_1 \cup F_2) + 1$, hence, $g(S'') = g(S') + g(F) = g(M', F_1 \cup F_2)$. Then S'' is a ∂-stabilization of S'. Hence, there are two cases.

Firstly, we may assume $V'' \cup_{S''} W''$ is a ∂-stabilization of $V' \cup_{S'} W'$ along F_2, where F_2 lies in W'.

It is easy to see that $V'' \cup_{S''} W''$ is weakly reducible. $V'' \cup_{S''} W''$ can be viewed as the amalgamation of $V' \cup_{S'} W'$ and $F \times I$. Then M can be decomposed as in Fig. 2.

The other case is that $V'' \cup_{S''} W''$ is a ∂-stabilization of $V' \cup_{S'} W'$ along F_1, where F_1 lies in V'. It can be found that M can also be decomposed as in Fig. 2.

By Lemma 2, $V' \cup_{S'} W'$ is unique up to isotopy. By [12], the Heegaard splitting of $F \times I$ is standard. Then the minimal Heegaard splitting of M is unique up to isotopy.

Case 2. F_1 and F_2 lie in the same side of S'.

Let $V \cup_S W$ be a minimal Heegaard splitting of M. We may assume F_1 and F_2 lie in W'. By the proof of Case 1, $V \cup_S W$ is weakly reducible.

By [11], $M = V \cup_S W = (V_1 \cup_{P_1} W_1) \cup_{H_1} \cdots \cup_{H_{n-1}} (V_n \cup_{P_n} W_n)$, where each $V_i \cup_{P_i} W_i$ is strongly irreducible, each H_i is incompressible. By the proof of Case 1, each component of H_i is parallel to $F_1 \cup F_2$ for all i. Let $H_1 = F_1 \cup F_2$, $M_1 = F \times I = V_1 \cup_{P_1} W_1$, and $V'' \cup_{S''} W'' = M'$ is the amalgamation of $(V_2 \cup_{P_2} W_2) \cup_{H_2} \cdots \cup_{H_{n-1}} (V_n \cup_{P_n} W_n)$. Then we have $g(S) = g(S'') + 1$. Since $g(S) \leqslant g(S') + 1$, $g(S'') \leqslant g(S')$. Since $d(S') > 2(g(M', F_1 \cup F_2) + 1) \geqslant 2g(S) > 2g(S'')$, by Lemma 5, S'' is isotopic to S'. Hence, M can be decomposed as in Fig. 3.

By Lemma 2, $V' \cup_{S'} W'$ is unique up to isotopy. By [12], the Heegaard splitting of $F \times I$ is standard. Hence, the minimal Heegaard splitting of M is unique up to isotopy. □

4. Example for $g(M) \leqslant g(M', F_1 \cup F_2)$

If M is the amalgamated 3-manifold of M_1 and M_2 along a separating surface F, then there is a natural Heegaard splitting of M. From this construction, we have $g(M) \leqslant g(M_1) + g(M_2) - g(F)$. There are two examples for $g(M) < g(M_1) +$

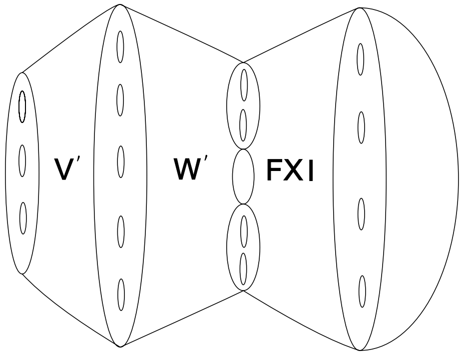

Fig. 3.

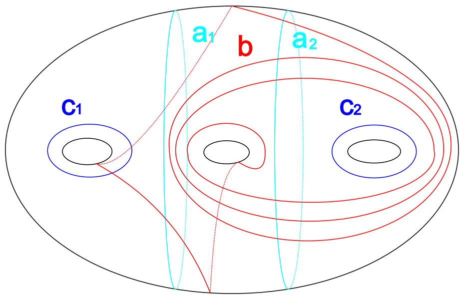

Fig. 4.

$g(M_2) - g(F)$ given by Kobayashi, Qiu, Rieck, Wang and Schultens, Weidmann, respectively. See [5] and [15]. Here, we give a simple example for the degeneration of Heegaard genus of self-amalgamated 3-manifold.

Let M be a 3-manifold, and A be an incompressible annulus on ∂M. Let $M = V \cup_S W$ be a Heegaard splitting with $A \subset \partial_- W$. Recall that a spine annulus in W is an essential annulus which one boundary component lies in $\partial_- W$, the other lies in $\partial_+ W$. A spine annulus A_s of W is called an A-spine annulus if one component of ∂A_s lies in A. $V \cup_S W$ is said to be A-primitive if there is an essential disk in V which intersects an A-spine annulus of W in one point. A 3-manifold M is said to be A-primitive if one of the minimal Heegaard splittings of M is A-primitive. Note that the two examples in [5] and [15] are both A-primitive.

Now we construct a 3-manifold M with $g(M) \leqslant g(M', F_1 \cup F_2)$. Let S' be a closed surface with $g(S') = 3$. See Fig. 4 for a Heegaard diagram: $\{a_1, a_2\}$ and $\{b\}$. Suppose W' is a compression body such that a_1 and a_2 bound essential disks in W', V' is a compression body such that b bounds an essential disk in V'. Then $M' = V' \cup_{S'} W'$ is strongly irreducible. Note that $\partial_- W'$ has three torus components T_1, T_2 and T_3, $\partial_- V'$ has only one component with $g(\partial_- V') = 2$. Suppose $c_i \subset T_i$. See Fig. 4. It is easy to see that $g(S')$ is minimal Heegaard genus of M'. Gluing T_1 to T_2 such that $c_1 = c_2$. We obtain a 3-manifold $M = M' \cup F \times I$, where $F \cong T_1 \cong T_2$. Since $|b \cap c_1| = 1$ and $c_1 \times I$ is a vertical annulus in W', $V' \cup_{S'} W'$ is A-primitive. When we glue T_1 to T_2, c_1 and c_2 bounds an annulus A in $W' \cup F \times I$. Let r be the spanning arc in A, $V = V' \cup N(r)$ and $W = M - \eta(r)$. Then $V \cup W$ is the natural Heegaard splitting of M. Note that $|b \cap A| = 1$, $A - \eta(r)$ is an essential disk D_W in W, b bounds an essential disk D_V in V. Since $b \cap c_2 = \emptyset$, $|D_V \cap D_W| = 1$. Hence, $V \cup W$ is stabilized, $g(M) \leqslant g(M', F_1 \cup F_2)$.

Remark 1. By Theorem 1.6 in [7], we can add a genus two handlebody and a solid torus to M', we denoted it by M''. If the gluing map is sufficiently complicated, S' is still a minimal Heegaard surface of M''. Hence, we can also construct a closed 3-manifold M with $g(M) \leqslant g(M', F_1 \cup F_2)$.

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